Constants of motion in a minimum-*B* mirror magnetic field

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A complete description of the particle motion in a three-dimensional marginal minimum-*B* mirror field is obtained, including two new constants of motion. The energy and the two Clebsch coordinates of the gyrocenter motion are constants of motion, and the magnetic moment is an adiabatic invariant. The invariance of the gyrocenter Clebsch coordinates implies that each gyrocenter bounces back and forth on a single magnetic field line. Complete solutions of the Vlasov equation can be constructed in the equilibrium field. A small gyroradius expansion of the Clebsch coordinate invariants splits the distribution function into a gyroaveraged part and a new gyro-oscillating part that gives rise to perpendicular plasma current. Locally omnigenous Vlasov equilibria to the first order in the plasma β can be constructed by including the diamagnetic drift. More than three time-independent invariants are required to obtain the general solution of the stationary Vlasov equation.

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I. INTRODUCTION

Particle motion in magnetic traps involves several key problems [1], such as the question of the existence of drift surfaces and the resonant interactions in microinstabilities and radio-frequency heating. A study of the particle motion is simplified when symmetry implies constancy of a generalized momentum, but a complete set of cyclic coordinates does not exist in three-dimensional mirror fields. Despite this, two simple constants of motions are found here for a marginal minimum-*B* mirror field.

Confinement of charges in a magnetic field is of interest in many branches of physics. A common topology is the toroidal geometry, but particles can also be confined in an open magnetic geometry in a region between magnetic maxima. Particles with large enough perpendicular velocity are reflected by the magnetic mirror. This phenomenon is exploited in the simple mirror trap [2], the Budker-Post mirror trap. Mirror trapping of charges appear around the earths magnetic field, and simple magnetic mirrors are common in laboratory experiments where reliable single particle confinement is crucial. Several experiments have confirmed that particles can be confined in magnetic mirror traps for times exceeding years. Although single-particle confinement appears to be ideal, plasma confinement is more complex and collisions, plasma instabilities, and resonant cyclotron frequency heating can push particles out from the mirror trap.

The first demand on the confining field is to provide stability toward large-scale magnetohydrodynamic (MHD) modes. In mirrors, a minimum-*B* magnetic field has been demonstrated both theoretically and experimentally to be sufficient for MHD stable plasma confinement [2]. The stability can be expected since the magnetic well of a minimum-*B* mirror field provides an increased magnetic field pressure in all directions. MHD stable confinement is achieved for plasma β values close to unity. The plasma β is the ratio of the plasma pressure to the magnetic field pressure $B^2/2\mu_0$, and a fusion reactor based on magnetic plasma confinement would require a β value exceeding 10% or so to provide an energy gain factor (the ratio of produced fusion power to the input power required to sustain the plasma) exceeding 10.

Superimposing a multipole field on the field produced by the mirror coils can create a minimum-*B* field. A drawback is that the multipole field gives a highly elliptic cross section of the magnetic flux surface near the mirrors. The optimal choice for a MHD stable mirror field from this point of view should be a *marginal* minimum-*B* field. For a plasma confined in a long and thin mirror trap, the unique solution for the vacuum magnetic field is derived in [3,4]. The optimal field corresponds to a "straight field line mirror" where the magnetic field lines are straight but nonparallel in the confining region. The straight field line mirror is a peculiar configuration with unique properties. Owing to them it may be useful in different applications.

In the directions perpendicular to the magnetic field, the charged particles gyrate around the magnetic field line of the gyrocenter, and a slow perpendicular gyrocenter drift is superimposed on this fast gyromotion. To first order in the ratio of the gyroradius to the gradient scale length of the electromagnetic fields, the perpendicular gyrocenter drift velocity is determined by

$$\mathbf{v}_{d,\perp} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mu}{q}\frac{\mathbf{B}\times\nabla B}{B^2} + \frac{m\mathrm{v}_{\parallel}^2}{qB^4}\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}],$$

where $\mu = m\mathrm{v}_{\perp}^2/2B$ is the magnetic moment of the gyrating particle. A second demand on plasma confinement is that the drifts in the radial direction out from the flux surfaces should vanish on average. The gyrocenter moves along an orbit that

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is nearly tangential to the flux surface, but a small "banana orbit width" associated with radial excursions from the magnetic surfaces appear in most confining fields. If the radial drift is zero and the drift surface lies on a flux surface, the field is said to be locally omnigenous in the confining region [5,6]. No locally omnigenous fields are known for toroidal devices, but tokamak magnetic fields are quasiomnigenous in the sense that the "mean drift surface" (the average over the radial banana excursions") lies on a flux surface. The banana widths can exceed the gyroradius by a large factor, providing increased radial steps in collisions events and thereby an enhanced radial leakage of the plasma from the confining field as predicted by the neoclassical transport theory. An interesting property with the "straight field line mirror" is that each gyrocenter moves along a single magnetic field line, and the neoclassical enhancement of the radial loss is thereby zero for this unique field.

Constants of motion are basic tools to analyze single particle and plasma properties derivable from the collisionless Boltzmann equation, i.e., the Vlasov equation. Jean's theorem states that an arbitrary function of the motional invariants is a solution of the Vlasov equation. Apart from exceptional cases involving global symmetries, a complete set of motional invariants has not been found for any realistic confining magnetic field. In this paper, we will show that a complete set of invariants can be constructed for the straight field line mirror, and this is a realistic three-dimensional field where this has been achieved. In addition to its relevance for plasma confinement, the straight field line mirror field could be instructive as an example of a nontrivial case with closed-form expressions for the motional invariants.

Because of the complexity involved in deriving expressions for a three-dimensional field and thereafter finding constants of motions, single particle studies for plasmas are often limited to the drift motion and designs of fields providing mean drift surfaces that nearly lie on the magnetic flux surfaces. For the straight field line mirror, we will prove, by applying the motional invariants derived in this paper to the collisionless Boltzmann equation, that the field is locally omnigenous to the first order in the plasma β. This means that each gyrocenter moves on a single flux surface even when the plasma currents are included, and there should be no neoclassical increase of the radial losses with this mirror magnetic field.

Additional critical requirements for a fusion reactor based on a magnetic mirror trap are control of plasma microinstabilities and plugging of end losses induced by these instabilities and collisions. A possibility to achieve this in a straight field line mirror by a "modified thermal barrier" is briefly analyzed theoretically in [7] and numerically in [8].

Catto and Hazeltine [6] have derived criteria from macroscopic fluid and gyrocenter drift equations for the existence of locally omnigenous mirror equilibria. Such equilibria are possible, for instance, if the plasma current has no component along the magnetic field [6].

Ryutov and Stupakov [9] have studied the problem of finite banana orbit widths by analyzing the gyroaveraged Vlasov equation and the longitudinal invariant. A velocity integration over the gyroaveraged distribution function gives no current, and when the gyrating part of the distribution function is unknown, the current can only be determined in an indirect manner from the general momentum balance, which involves gradients of the gyroaveraged pressure tensor and the condition of a divergence free current [9].

II. MAGNETIC FIELD AND FLUX COORDINATES

In this paper, we will show how the gyrating part of the distribution function and the associated diamagnetic current can be found by using the complete set of motional invariants in the marginal minimum-B field. All the invariants are found to be even functions of the parallel velocity, and no parallel plasma current exists in this field. A plasma confined in such a vacuum field would therefore give a locally omnigenous equilibrium to the first order in the plasma β, if some mild restrictions are satisfied by the pressure tensor components [2].

To show this, we use the expression, derived in Refs. [3,4] in the near paraxial limit, for a marginal minimum-B vacuum field,

$$\mathbf{B}_{\rm v} = B_{\rm v}(s)\, \nabla s + O\left(\frac{a^4}{c^4}\right), \tag{1}$$

where $B_{\rm v}(s)=B_0/(1-s^2/c^2)$, s is the arc length along the magnetic field lines, the constant B_0 is the magnetic field strength at the central surface $s=0$, c is the characteristic longitudinal length scale, and a is the central surface flux tube radius. No particles are able to transit the infinite field at $s=\pm c$. A finite region $|s|<s_1$, where $s_1<c$, can be defined to which the particle motion is bounded by stipulating conditions on the pitch angle at the central surface (cf. Ref. [10]). The infinite model field in the outer region is of no real concern, since it has been shown in Ref. [4] how a realistic field can be constructed that is bounded at the outer region and practically identical to the model field in the confinement region $|s|<s_1$.

The error for the vacuum field is insignificant for a long and thin flux tube, since the representative parameter range $a/c<5\%$ gives $a^4/c^4<10^{-5}$. Neglecting the correction term, we may write

$$\frac{\mathbf{B}_{\rm v}}{B_0} = \nabla x_0 \times \nabla y_0 = \frac{\nabla s}{1-s^2/c^2}, \tag{2}$$

where (x_0,y_0) are Cartesian-like Clebsch coordinates. A flux coordinate system is introduced by the transformation $(x,y,z)\to(x_0,y_0,s)$. In Ref. [4], expressions for the flux coordinates have been derived

$$s(x,y,z) = z + \frac{c}{2}\left(\frac{x^2/c^2}{1+\bar{z}} - \frac{y^2/c^2}{1-\bar{z}}\right) + O\left(\frac{a^4}{c^4}\right), \tag{3}$$

$$x_0(x,y,z) = \frac{x}{1+\bar{s}}\left[1+\frac{1}{6}\left(\frac{x/c}{1+\bar{s}}\right)^2\right] + O\left(\frac{a^5}{c^5}\right), \tag{4}$$

$$y_0(x,y,z) = \frac{y}{1-\bar{s}}\left[1+\frac{1}{6}\left(\frac{y/c}{1-\bar{s}}\right)^2\right] + O\left(\frac{a^5}{c^5}\right), \tag{5}$$

where $\bar{z}=z/c$ and $\bar{s}=s/c$. Equation (3) shows that the surface $s=0$ is slightly curved, since at $s=0$ we obtain

$\bar{z} \approx -(x^2 - y^2)/(2c^2)$. Extending the analysis to more general fields would be a complex procedure, since analytical expressions for the flux coordinates cannot be obtained analytically other than in exceptional cases. For any given field $B(x,y,z)$, numerical ray tracing along the field lines from a prescribed surface $s(x,y,z)=0$ is the standard means to generate flux coordinates.

The scale factors in terms of the coordinates (x_0, y_0, s) are $|\nabla x_0| = (1+\bar{s})^{-1}$, $|\nabla y_0| = (1-\bar{s})^{-1}$ and $\nabla x_0 \cdot \nabla y_0 = -(x_0 y_0/c^2) \times (1-\bar{s}^2)^{-1}$, where the corrections are $O(a^4/c^4)$. The x_0 and y_0 coordinates are stretched or contracted as $B_{\mathrm{v}}(s)$ increases away from its minimum at $s=0$. We also have $|\nabla s| = 1 + O(a^4/c^4)$ and $\nabla s \approx \hat{\mathbf{B}}_{\mathrm{v}}$ is a unit vector along the vacuum magnetic field. In addition, x_0 and y_0 are constant along $\mathbf{B}_{\mathrm{v}}$ since $\nabla x_0 \cdot \mathbf{B}_{\mathrm{v}} = \nabla y_0 \cdot \mathbf{B}_{\mathrm{v}} = 0$.

The flux lines corresponding to Eqs. (3)–(5) can to the leading order be parametrized as

$$x(s) = (1+\bar{s})x_0 \approx (1+\bar{z})x_0, \tag{6}$$

$$y(s) = (1-\bar{s})y_0 \approx (1-\bar{z})y_0. \tag{7}$$

This corresponds to straight (nonparallel) flux lines in the confining region with "focal lines" at $z = \pm c$ (cf. [4]). With a circular flux tube cross section at the central surface, the flux tube boundary is determined by

$$a^2 = x_0^2 + y_0^2 = \left(\frac{x}{1+\bar{s}}\right)^2 + \left(\frac{y}{1-\bar{s}}\right)^2 \tag{8}$$

and the local ellipticity is $\varepsilon_{ell}(s) = (1+|\bar{s}|)/(1-|\bar{s}|) = (\sqrt{R_m} + \sqrt{R_m - 1})^2$, where $R_m(s) = B_{\mathrm{v}}(s)/B_0$ is the local mirror ratio. The maximum ellipticity in the confinement region appears at the mirrors, where the magnetic field strength has the largest magnitude in the confinement region. Since it is plausible that a marginal minimum-*B* field gives the optimal ellipticity [4], this expression may determine the smallest possible ellipticity at the mirrors for a MHD stable confinement.

Another important property with the marginal minimum-*B* vacuum field is that the gyrocenter drift formula shows that the x_0 and y_0 gyrocenter coordinates are constant during the motion, since the perpendicular gyrocenter drift motion is zero [3]. Each gyrocenter moves on a *single* field line, as pointed out in Ref. [3].

III. NEW MOTIONAL INVARIANTS

To include the Larmor circles, we use that the Lagrangian $L = m\mathrm{v}^2/2 - q\phi(s) + q\mathbf{v}\cdot\mathbf{A}$ is invariant under point transformations, and it is here assumed that the electric potential only depends on the arc lengths s. We choose the vector potential

$$\mathbf{A} = \frac{B_0}{2}(x_0 \nabla y_0 - y_0 \nabla x_0). \tag{9}$$

To be able to express the Lagrangian in the (x_0, y_0, s) coordinates, we use

$$\frac{\partial \mathbf{x}}{\partial x_0} \equiv \frac{\nabla y_0 \times \nabla s}{\nabla s \cdot (\nabla x_0 \times \nabla y_0)} = (1-\bar{s}^2)\, \nabla y_0 \times \hat{\mathbf{B}}_{\mathrm{v}}, \tag{10}$$

$$\frac{\partial \mathbf{x}}{\partial y_0} \equiv -\frac{\nabla x_0 \times \nabla s}{\nabla s \cdot (\nabla x_0 \times \nabla y_0)} = -(1-\bar{s}^2)\, \nabla x_0 \times \hat{\mathbf{B}}_{\mathrm{v}}, \tag{11}$$

$$\frac{\partial \mathbf{x}}{\partial s} \equiv \frac{\nabla x_0 \times \nabla y_0}{\nabla s \cdot (\nabla x_0 \times \nabla y_0)} = \hat{\mathbf{B}}_{\mathrm{v}}. \tag{12}$$

The velocity is determined by

$$\mathbf{v} \equiv \frac{d\mathbf{x}}{dt} = \dot{s}\hat{\mathbf{B}}_{\mathrm{v}} + \dot{x}_0 \frac{\partial \mathbf{x}}{\partial x_0} + \dot{y}_0 \frac{\partial \mathbf{x}}{\partial y_0} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}, \tag{13}$$

where $\dot{s} \equiv ds/dt$, $\mathbf{v}_{\parallel} = \dot{s}\hat{\mathbf{B}}_{\mathrm{v}}$ and

$$\mathbf{v}_{\perp} = (1-\bar{s}^2)(\dot{x}_0 \nabla y_0 - \dot{y}_0 \nabla x_0) \times \hat{\mathbf{B}}_{\mathrm{v}}. \tag{14}$$

From this follows $q\mathbf{v}\cdot\mathbf{A} = (-\dot{x}_0 y_0 + \dot{y}_0 x_0)m\Omega_0/2$, where $\Omega_0 = qB_0/m$. Errors of order $\mathrm{O}(a^2/c^2)$ are insignificant for a long and thin mirror flux tube. The Lagrangian may thus be approximated by

$$\frac{L}{m} = -\frac{q}{m}\phi(s) + \frac{\Omega_0}{2}(-\dot{x}_0 y_0 + \dot{y}_0 x_0) + \frac{\dot{s}^2}{2} + \frac{(1+\bar{s})^2 \dot{x}_0^2}{2} + \frac{(1-\bar{s})^2 \dot{y}_0^2}{2}. \tag{15}$$

For the perpendicular motion, the Lagrange equations for the (x_0, y_0) coordinates give two first integrals:

$$(1+\bar{s})^2 \dot{x}_0 - \Omega_0 y_0 = -\Omega_0 I_y, \tag{16}$$

$$(1-\bar{s})^2 \dot{y}_0 + \Omega_0 x_0 = \Omega_0 I_x. \tag{17}$$

Although the Lagrangian (15) gives exactly constant values of I_x and I_y, a more accurate derivation, including terms of order $\mathrm{O}(a^2/c^2)$ in the Lagangian, would give a slow time dependence for I_x and I_y; but this is expected to be insignificant for sufficiently small values of a/c. A check shows that the Poisson bracket $\{I_y, I_x\}_{\mathbf{q},\mathbf{p}} = 1/\Omega_0 \neq 0$, i.e., the invariants are not in involution, and this implies that these invariants cannot be connected with a pair of cyclic coordinates in the Hamilton-Jacobi theory. We will show that the constants of motion I_x and I_y are the guiding center values of the (x_0, y_0) coordinates, and this explains why there is no perpendicular drift of the gyrocenter motion. After introducing new time variables by $dt_\sigma/dt = (1+\sigma\bar{s})^{-2}$, where $\sigma = \pm$, we obtain by combining the pair of equations for the invariants

$$\frac{d^2 x_0}{dt_+^2} + (x_0 - I_x)\Omega_0^2 \frac{(1+\bar{s})^2}{(1-\bar{s})^2} = 0, \tag{18}$$

$$\frac{d^2 y_0}{dt_-^2} + (y_0 - I_y)\Omega_0^2 \frac{(1-\bar{s})^2}{(1+\bar{s})^2} = 0. \tag{19}$$

Particular solutions are simply the constants $x_0 = I_x$ and $y_0 = I_y$, which can be identified as the guiding center values, since the homogenous part of the equations describe oscillatory motions. Since $a/c \ll 1$, the gyrofrequency is slowly varying and a WKB (Wentzel-Kramers-Brillouin) approximation gives to the leading order

$$x_0 = I_x + \frac{\sqrt{2\mu B_v(s)/m}}{\Omega_0}(1-\bar{s})\cos\varphi_g, \tag{20}$$

$$y_0 = I_y - \frac{\sqrt{2\mu B_v(s)/m}}{\Omega_0}(1+\bar{s})\sin\varphi_g, \tag{21}$$

where $\varphi_g=\varphi_{g,0}+\int_0^t\Omega[s(t')]dt'$ is the gyroangle, the constant $\varphi_{g,0}$ determines the gyrophase, and $\Omega(s)=\Omega_0/(1-\bar{s}^2)$ is the local gyrofrequency. The constant μ can be interpreted from

$$\mu \approx \frac{1}{B_v(s)}\frac{m}{2}[(1+\bar{s})^2\dot{x}_0^2+(1-\bar{s})^2\dot{y}_0^2] = \frac{m\mathrm{v}_\perp^2/2}{B_\mathrm{v}(s)}. \tag{22}$$

Thus, μ is the adiabatic invariant (the magnetic moment), and μ is to the leading order proportional to the ratio of the perpendicular kinetic energy to the slowly varying gyrofrequency.

The motion along the magnetic field is constrained by the constancy of the energy ε,

$$\dot{s} = \pm\sqrt{\frac{2}{m}[\varepsilon - q\phi(s)] - (1+\bar{s})^2\dot{x}_0^2 - (1-\bar{s})^2\dot{y}_0^2}. \tag{23}$$

The relation $m\mathrm{v}_\perp^2/2 \approx \mu B_\mathrm{v}(s)$ implies that the leading-order parallel motion can be solved independently of the perpendicular motion. In this standard gyroaveraged approximation, the periodic parallel motion $s(\tau_\parallel)$ is obtained by inverting the formula

$$\tau_\parallel(s) - \tau_\parallel(s_0) = \pm\int_{s_0}^{s}\frac{ds}{\sqrt{\frac{2}{m}}\sqrt{\varepsilon - q\phi(s) - \frac{\mu B_0}{1-\bar{s}^2}}}, \tag{24}$$

where the constant $\tau_\parallel(s_0)$ determines the phase of the motion along the field line. In case $\phi(s)=0$, $\tau_\parallel(s)$ can be expressed in terms of an elliptic integral [10].

IV. LOCALLY OMNIGENUOS VLASOV EQULIBRIA WITH FINITE β

Let us proceed to show how the diamagnetic current can be determined with the help of the gyrocenter Clebsch coordinate invariants. We illustrate the procedure by considering solutions of the Vlasov equation of the form

$$F = n_0(I_x,I_y)\bar{F}(\varepsilon,\mu) = F_c + F_g, \tag{25}$$

where $n_0(x_0,y_0)$ can be identified with the density at $s=0$, provided a proper normalization of $\bar{F}(\varepsilon,\mu)$ is chosen. If the gyroradius is small compared to the gradient scale length of $n_0(x_0,y_0)$, a Taylor expansion around $(I_x,I_y)=(x_0,y_0)$ gives

$$n_0(I_x,I_y) = n_0(x_0,y_0) + \frac{\partial n_0}{\partial x_0}\frac{(1-\bar{s})^2\dot{y}_0}{\Omega_0} - \frac{\partial n_0}{\partial y_0}\frac{(1+\bar{s})^2\dot{x}_0}{\Omega_0}. \tag{26}$$

This results in $F_c=n_0(x_0,y_0)\bar{F}(\varepsilon,\mu)$ for the gyroaveraged part, which is identical to the form used in Ref. [10]. This expression requires that the (x_0,y_0) gyrocenter coordinates are constant during the motion, but this property is not satisfied in arbitrary mirror fields. One possible approach for those kinds of mirror fields is to choose a gyrocenter distribution function of the form $F_c(J_\parallel,\varepsilon,\mu)$, provided the longitudinal action $J_\parallel(x_0,y_0,\varepsilon,\mu)$ is invariant, as was done in Ref. [9]. The perpendicular pressure tensor component is

$$P_\perp \equiv \int d^3\mathrm{v}\frac{m\mathrm{v}_\perp^2}{2}F = \int_{-\infty}^{\infty}d\mathrm{v}_\parallel\int_0^{\infty}d\mathrm{v}_\perp\mathrm{v}_\perp\oint d\varphi_g\frac{m\mathrm{v}_\perp^2}{2}F_c. \tag{27}$$

Although the gyrating part of Eq. (25), i.e.,

$$F_g = \left[-\frac{\partial n_0}{\partial x_0}\frac{(1-\bar{s})\mathrm{v}_\perp\cos\varphi_g}{\Omega_0} + \frac{\partial n_0}{\partial y_0}\frac{(1+\bar{s})\mathrm{v}_\perp\sin\varphi_g}{\Omega_0}\right]\bar{F}(\varepsilon,\mu), \tag{28}$$

gives no contribution to the density and the pressure tensor components, a finite plasma current can be calculated from $\mathbf{j}=\int q\mathbf{v}F_g d^3\mathrm{v}$, with the result $\mathbf{j}=-\boldsymbol{\nabla}P_\perp\times\hat{\mathbf{B}}_\mathrm{v}/B_\mathrm{v}$. As expected, this gives the momentum balance $\mathbf{j}\times\mathbf{B}_\mathrm{v}=\boldsymbol{\nabla}_\perp P_\perp$ (cf. Ref. [9]). Since $\mathbf{j}_\parallel=0$, locally omnigenous Vlasov equilibria to the first order in the plasma β can be constructed [6]. A sufficient criterion is that $P_\perp$ is independent of the Clebsch angle coordinate [6], which is satisfied if $n_0=n_0(\sqrt{I_x^2+I_y^2})$. Finally, with $\beta=2\mu_0P_\perp/B_\mathrm{v}^2(s)$, the plasma currents gives rise to the magnetic field

$$\mathbf{B}_{pl} = -\frac{\beta}{2}\frac{B_0}{1-\bar{s}^2}\boldsymbol{\nabla}s + \boldsymbol{\nabla}\phi_{m,pl}, \tag{29}$$

where the condition $\boldsymbol{\nabla}\cdot\mathbf{B}_\mathrm{pl}=0$ is satisfied if $\phi_{m,pl}$ is given by the Coulomb integral

$$\phi_{m,pl}(\mathbf{x}) = -\frac{B_0}{8\pi}\int\frac{dV'}{1-\bar{s}'^2}\frac{\partial\beta/\partial s'}{|\mathbf{x}-\mathbf{x}'|}. \tag{30}$$

V. THE HAMILTON-JACOBI EQUATION

The Hamilton-Jacobi equation with a generating function $G(\mathbf{q},\mathbf{P})$ of the old canonical coordinates, and the new momenta reads for the "straight field line mirror"

$$2m\varepsilon = \left(\frac{\partial G}{\partial s}\right)^2 + \frac{\left(\frac{\partial G}{\partial x_0} + y_0\frac{m\Omega_0}{2}\right)^2}{(1+\bar{s})^2} + \frac{\left(\frac{\partial G}{\partial y_0} - x_0\frac{m\Omega_0}{2}\right)^2}{(1-\bar{s})^2}. \tag{31}$$

We select for the new coordinates $Q_1=Q_1(I_x,I_y)$ as an arbitrary function of the Clebsch coordinate invariants, $Q_2=\varphi_g(t)$ is the gyroangle and $Q_3=s$ is the arc length along $\mathbf{B}$. We choose a generating function independent of Q_1 and P_1

$$G(\mathbf{q},\mathbf{P}) = sP_3 + \Omega(s)P_2 + \frac{\sqrt{3}}{4}m\Omega_0\left(\frac{1+\bar{s}}{1-\bar{s}}x_{0,g}^2 + \frac{1-\bar{s}}{1+\bar{s}}y_{0,g}^2\right) - \frac{m\Omega_0}{2}(x_0I_y - y_0I_x),$$

where $x_{0,g}=x_0-I_x$ and $y_{0,g}=y_0-I_y$, and (I_x,I_y) are integration constants (not canonical variables). Within the accuracy of the WKB approximation in Eqs. (20) and (21) (which gives an exact solution in the limit of a constant magnetic field, i.e., $c \to \infty$), this gives the result $\varepsilon = \mu B(s) + m\dot{s}^2/2 + O(a/c)$ for the guiding center motion, compare [11] or [12] where a Hamilton-Jacobi treatment of RF heating is carried out. The Hamiltonian in the new canonical coordinates reads

$$K(\mathbf{Q},\mathbf{P}) = \frac{P_3^2}{2m} + \Omega(Q_3)P_2,$$

where $P_2 = m\mu/q$ is proportional to the conserved magnetic moment. Since this Hamiltonian is independent of both Q_1 and P_1, these canonical variables are arbitrary functions of the motional invariants. A convenient choice, whereby the phase space can be spanned by the canonical variables, is $Q_1 = I_x$ and $P_1 = I_y$. It is straightforward to make a subsequent canonical transformation with the bounce time $\tau_{\parallel}(s)$ as a new canonical coordinate, to obtain a Hamiltonian that is independent on the coordinates, whereby all momenta in this set of canonical variables are constant during the motion.

A "quasiparadox" is that, at most, *three* time-independent invariants are typically expected from the Hamilton-Jacoby equation, but it is obvious from this calculations that *four* time-independent invariants exist in certain systems, in particular if there is a degeneracy in the frequencies. This is not in contradiction with the theorem by Liouville that states if three constants of motions in involution are found, the equations of motion for a point mass can be integrated by quadratures, see [13]. Although three invariants are sufficient to integrate the motion, it needs to be stressed that the general solution of the stationary collisionless Boltzmann equation is an arbitrary function of *all* stationary invariants.

VI. CONCLUSION

The invariance of the gyrocenter flux coordinates gives two different constants of motion I_x and I_y, defined in Eqs. (16) and (17), for a marginal minimum-*B* field. This pair of invariants, as well as the energy and magnetic moment, are even functions of the parallel velocity, and, thus, $\mathbf{j}_{\parallel} = \mathbf{0}$ for any distribution function of the form $F(I_x, I_y, \varepsilon, \mu)$. Locally omnigenous equilibria to the first order in the plasma beta are derived in this paper, and a closed-form expression for a finite β magnetic field is determined. This minimum-*B* magnetic field provides MHD stability, even for a finite β (cf. Ref. [14]) and gives well-defined drift surfaces with respect to ∇B and curvature drifts. This minimum-*B* field may also correspond to the minimal ellipticity at the mirrors.

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